

CONVERGENCE RATE OF STABILITY PROBLEMS OF SDES WITH (DIS-)CONTINUOUS COEFFICIENTS

HIROYA HASHIMOTO AND TAKAHIRO TSUCHIYA

ABSTRACT. We consider the stability problems of one dimensional SDEs when the diffusion coefficients satisfy the so called Nakao-Le Gall condition. The explicit rate of convergence of the stability problems are given by the Yamada-Watanabe method without the drifts. We also discuss the convergence rate for the SDEs driven by the symmetric α stable process. These stability rate problems are extended to the case where the drift coefficients are bounded and in L^1 . It is shown that the convergence rate is invariant under the removal of drift method for the SDEs driven by the Wiener process.

1. INTRODUCTION

Consider the following sequence of one-dimensional stochastic differential equations, SDEs for short,

$$(1) \quad X_n(t) = X_n(0) + \int_0^t b_n(X_n(s))ds + \int_0^t \sigma_n(X_n(s))dW_s,$$

and consider the solution X given by

$$(2) \quad X(t) = X(0) + \int_0^t b(X(s))ds + \int_0^t \sigma(X(s))dW_s,$$

where $\{W_s\}_{s \geq 0}$ is a Wiener process and $b_n : \mathbb{R} \rightarrow \mathbb{R}$ and $\sigma_n : \mathbb{R} \rightarrow \mathbb{R}$ for $n \in \mathbb{N}$ are coefficients which tend to b and σ respectively in some sense, as $n \rightarrow \infty$. The convergence of the sequence $\{X_n\}_{n \in \mathbb{N}}$ to X named as stability problems was introduced by Stroock and Varadhan in [12] to solve the martingale problems for unbounded coefficients b and σ . The stability problem in the *strong* sense was treated by Kawabata-Yamada [8] in the case where the diffusion coefficients are Hölder continuous of exponent $\alpha \geq 1/2$. Le Gall investigated the stability problems, when the diffusion coefficients are positive and squared finite quadratic variation in [9]. Kaneko and Nakao showed in [7] that the pathwise uniqueness implies the stability property if the coefficients satisfy some condition on the modulus continuity or if the diffusion coefficient is positive definite.

The rate of convergence of the Euler-Maruyama scheme to the solution has been discussed by Deelstra and Delbaen in [3] and their results are considerably generalized by Gyöngy and Rásonyi in [5]. In their investigations, the Yamada-Watanabe method introduced in [14], [17] and [16] plays essential roles. Gyöngy and Rásonyi in [5] obtained the rate of convergence in the case where the diffusion coefficients are $(1/2 + \gamma)$ -Hölder continuous with the suitable drift coefficients in L^1 . It is remarkable that the rate of convergence is given by $n^{-\gamma}$ where $\gamma > 0$ and also is given by $(\log n)^{-1}$ in the case where $\gamma = 0$.

These results suggest that the rate of convergence of the stability problems may depend also on the modulus continuity or irregularity of diffusion coefficients. Since the coefficient may be discontinuous under the Nakao-Le Gall condition, it seems to be very interesting to investigate the rate of convergence of the stability problems.

The present paper is organized as follows: Section 2 is devoted to the assumption and preliminaries. Our main result is shown in Section 3. Let us consider first drift-less system

$$(3) \quad X_n(t) = X_n(0) + \int_0^t \sigma_n(X_n(s)) dW_s,$$

$$(4) \quad X(t) = X(0) + \int_0^t \sigma(X(s)) dW_s,$$

for $t \geq 0$ and $X_n(0) \equiv X(0)$. These diffusion coefficients satisfy the so called Nakao-Le Gall condition (See Definition 2.1 below). Assume that there exists a positive constant C_0 such that

$$\int_{\mathbb{R}} |\sigma_n(x) - \sigma(x)| dx \leq C_0 n^{-1} \text{ or } \sup_{x \in \mathbb{R}} |\sigma_n(x) - \sigma(x)| \leq C_0 n^{-1},$$

for all $n \in \mathbb{N}$. It will be shown that there exist positive constants C_p ($p \geq 1$) such that

$$\mathbb{E}[|X(t) - X_n(t)|] \leq C_1 (\log n)^{-\frac{1}{2}} \quad (0 \leq t \leq T)$$

and

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} |X(t) - X_n(t)|^p \right] \leq C_p (\log n)^{-\frac{p}{4(p+1)}}$$

holds for $p > 1$ and $n > 2$. In Section 4, we also discuss the convergence rate of the stability problems in the case where the SDEs are driven by a symmetric α stable process without the drift for $\alpha \in (1, 2)$. Finally, the stability problems with the drift coefficients will be discussed using the removal of drift method in Section 5.

2. ASSUMPTIONS AND PRELIMINARIES

On a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with the filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual condition, let us consider the stochastic differential equations (1) and (2). For the simplify, we assume that $X(0) \equiv X_n(0)$. The following condition on the diffusion coefficients was proposed by Nakao [10] and modified by Le Gall [9].

Definition 2.1. *In this paper, we say that a real valued function σ satisfies the Nakao-Le Gall condition and write $\sigma \in \mathcal{C}_{NL}(\epsilon, f)$ if σ satisfies the following statements:*

- (1) *There exists a positive real number ϵ such that*

$$\epsilon \leq \sigma(x)$$

holds for any x in \mathbb{R} .

- (2) *There exists an increasing function f such that*

$$|\sigma(x) - \sigma(y)|^2 \leq |f(x) - f(y)|$$

holds for every x and y in \mathbb{R} .

In addition, if the function f is bounded on an interval $I \subset \mathbb{R}$ as follows:

$$\|f\|_{I, \infty} := \sup_{x \in I} |f(x)| < \infty,$$

then we denote $\sigma \in \mathcal{C}_{NL}(\epsilon, \|f\|_{I, \infty})$. In particular, in the case where $I = \mathbb{R}$, we write $\sigma \in \mathcal{C}_{NL}(\epsilon, \|f\|_{\infty})$ and $\|f\|_{\infty} := \|f\|_{\mathbb{R}, \infty}$.

Remark 2.1. *For the equation (2), if b is bounded measurable function and $\sigma \in \mathcal{C}_{NL}(\epsilon, \|f\|_{\infty})$, then there exists a unique strong solution X , see Theorem 1.3 (3) of the paper [9].*

Remark 2.2. *If the Nakao-Le Gall condition holds, $\sigma \in \mathcal{C}_{NL}(\epsilon, f)$, we can select a sequence of smooth functions $\{f_l\}_{l \in \mathbb{N}}$ such that for continuous points $x, y \in \mathbb{R}$,*

$$|\sigma(x) - \sigma(y)|^2 \leq |f_l(x) - f_l(y)| \text{ and } f_l(x) \uparrow f(x) \text{ (} l \rightarrow \infty \text{)}.$$

Then for an progressively measurable process X we have that

$$\int_0^t f(X(s))ds = \lim_{l \rightarrow \infty} \int_0^t f_l(X(s))ds$$

holds almost surely. We use this fact in the proof of Theorem 3.1 through 5.1.

Now, we shall introduce the Yamada-Watanabe method. Take a decreasing sequence $(a_m)_{m \in \mathbb{N}}$ of positive numbers satisfying $\infty > a_1 > \dots > a_m > \dots \downarrow 0$ and $\int_{a_m}^{a_{m-1}} x^{-1} dx = m$. Let us consider a smooth symmetric around the origin function $\varphi_m(\cdot)$ with support in $(-a_{m-1}, -a_m)$ and (a_m, a_{m-1}) such that $0 \leq \varphi_m(x) \leq 2(xm)^{-1}$ holds and $\int_{\mathbb{R}} \varphi_m(y)dy = 1$. For all $x \in \mathbb{R}$ define

$$u_m(x) = \int_0^{|x|} \int_0^y \varphi_m(z) dz dy.$$

Then we have

$$(5) \quad |x| \geq u_m(x) \geq -a_{m-1} + |x|, \quad 1 \geq |u'_m(x)|$$

and $u''_m(x) = \varphi_m(x)$ holds for $x \in \mathbb{R}$, see [16].

3. A STRONG CONVERGENCE RATE OF THE STABILITY PROBLEMS

In this section, we consider the convergence rate of the stability problems for the drift-less system (3) and (4) under the Nakao-Le Gall condition. The stability problem with the L^1 -convergence was discussed by Le Gall [9]. In more detail, the explicit rate of the convergence in L^p sup-norm is obtained as follows:

Theorem 3.1. *Let $T > 0$ and $p \geq 1$ and let X_n for $n \in \mathbb{N}$ be solutions of (3) and X be a solution of (4) such that $\mathbb{E}|X(0)|^p < \infty$ and $X_n(0) \equiv X(0)$. Suppose that σ and $\{\sigma_n\}_{n \in \mathbb{N}}$ satisfy the Nakao-Le Gall condition, i.e., $\sigma, \sigma_n \in \mathcal{C}_{NL}(\epsilon, \|f\|_\infty)$. Assume one of the following stability rate conditions holds: there exists a positive constant C_0 such that*

$$(6) \quad \int_{\mathbb{R}} |\sigma_n(x) - \sigma(x)|^2 dx \leq C_0 n^{-1}, \quad n \in \mathbb{N},$$

or

$$(7) \quad \sup_{x \in \mathbb{R}} |\sigma_n(x) - \sigma(x)| \leq C_0 n^{-1}, \quad n \in \mathbb{N}.$$

Then there exist positive constants C_p ($p \geq 1$) such that

$$\mathbb{E}[|X(t) - X_n(t)|] \leq C_1 (\log n)^{-\frac{1}{2}} \quad (0 \leq t \leq T)$$

and

$$(8) \quad \mathbb{E} \left[\sup_{0 \leq t \leq T} |X(t) - X_n(t)|^p \right] \leq C_p (\log n)^{-\frac{p}{4(p+1)}}$$

holds for $p > 1$ and $n > 2$.

Proof. The boundedness of σ and σ_n implies that

$$(9) \quad \mathbb{E} \left[\sup_{0 \leq t \leq T} |X(t)|^p \right] + \sup_{n \in \mathbb{N}} \mathbb{E} \left[\sup_{0 \leq t \leq T} |X_n(t)|^p \right] < \infty.$$

Define

$$Y_n(t) := X(t) - X_n(t).$$

Let u_m be a function as defined in Section 2. By (5) and then applying Itô's formula to $u_m(Y_n(\cdot))$

$$(10a) \quad \begin{aligned} |X(t) - X_n(t)| &\leq a_{m-1} + u_m(Y_n(t)) \\ &= a_{m-1} + M(t) + J(t), \end{aligned}$$

where M and J are defined as follows:

$$\begin{aligned} M(t) &= \int_0^t u'_m(Y_n(s)) \{ \sigma(X(s)) - \sigma_n(X_n(s)) \} dW_s, \\ J(t) &= \frac{1}{2} \int_0^t u''_m(Y_n(s)) | \sigma(X(s)) - \sigma_n(X_n(s)) |^2 ds. \end{aligned}$$

Now we also define J^σ and J^Y ,

$$\begin{aligned} (10b) \quad J(t) &\leq \int_0^t u''_m(Y_n(s)) | \sigma(X(s)) - \sigma_n(X(s)) |^2 ds \\ &\quad + \int_0^t u''_m(Y_n(s)) | \sigma_n(X(s)) - \sigma_n(X_n(s)) |^2 ds, \\ &=: J^\sigma(t) + J^Y(t), \text{ say for } 0 \leq t \leq T. \end{aligned}$$

Note that by (10b)

$$\begin{aligned} (10c) \quad |X(t) - X_n(t)|^p &\leq |a_{m-1} + |M(t)| + J^\sigma(t) + J^Y(t)|^p \\ &\leq 4^{p-1} \{ a_{m-1}^p + |M(t)|^p + |J^\sigma(t)|^p + |J^Y(t)|^p \}. \end{aligned}$$

Then taking the sup-norm over the time interval $[0, T]$, we have

$$\begin{aligned} (10d) \quad \sup_{0 \leq t \leq T} |X(t) - X_n(t)|^p &\leq 4^{p-1} \left\{ a_{m-1}^p + \sup_{0 \leq t \leq T} |M(t)|^p + |J^\sigma(T)|^p + |J^Y(T)|^p \right\}. \end{aligned}$$

Selecting the smooth functions f_l ($l \in \mathbb{N}$) in Remark 2.2, we have for $x, y \in \mathbb{R}$,

$$f_l(x) - f_l(y) = (x - y) \int_0^1 f'_l(x + \theta(y - x)) d\theta.$$

Then we define and consider J_l^Y as follows:

$$\begin{aligned} J_l^Y(t) &:= \int_0^t u''_m(Y_n(s)) |f_l(X(s)) - f_l(X_n(s))| ds \\ &= \int_0^t \varphi_m(Y_n(s)) |Y_n(s)| 1_{(0 < |Y_n(s)|)} \frac{|f_l(X(s)) - f_l(X_n(s))|}{|Y_n(s)|} ds \\ &\leq 2m^{-1} \int_0^t \int_0^1 f'_l(Z^\theta(s)) d\theta ds \end{aligned}$$

holds where for $\theta \in [0, 1]$ we define

$$Z(t) \equiv Z^\theta(t) := X(t) + \theta(X_n(t) - X(t)).$$

The fact that the martingale part of Z is expressed by

$$\int_0^t \{ (1 - \theta) \sigma(X(s)) + \theta \sigma_n(X_n(s)) \} dW_s =: \int_0^t \tilde{\sigma}(s) dW_s,$$

and $(1 - \theta) \sigma(x) + \theta \sigma_n(x) \geq \epsilon$ for $x \in \mathbb{R}$ implies that we have $\langle Z, Z \rangle_t \geq \epsilon^2 t$. Now the occupation times formula implies that we have

$$\int_0^t \int_0^1 f'_l(Z^\theta(s)) d\theta ds \leq \epsilon^{-2} \int_0^1 \int_{-\infty}^{\infty} f'_l(a) L_t^a(Z^\theta) da d\theta$$

where $L_t^a(Z)$ stands for the local time of Z accumulated at a until time t . By the Meyer-Tanaka formula and L^p integrability of the solutions X and X_n of (9), we have

$$(10e) \quad c_L := \sup_{\theta \in [0,1]} \sup_{a \in \mathbb{R}} \mathbb{E}[(L_T^a(Z^\theta))^p] < \infty.$$

Then we obtain

$$\begin{aligned} \mathbb{E}[|J_l^Y(t)|^p] &\leq 2^p(m\epsilon^2)^{-p} \mathbb{E}\left[\left(\int_0^1 \int_{-\infty}^{\infty} f'_l(a) L_t^a(Z^\theta) da d\theta\right)^p\right] \\ &\leq 2^p(m\epsilon^2)^{-p} \left(\int_0^1 \int_{-\infty}^{\infty} f'_l(a) da d\theta\right)^{p-1} \mathbb{E}\left[\int_0^1 \int_{-\infty}^{\infty} f'_l(a) (L_t^a(Z^\theta))^p da d\theta\right] \\ &\leq 2^p(m\epsilon^2)^{-p} \|f_l\|_\infty^{p-1} \int_0^1 \int_{-\infty}^{\infty} f'_l(a) \mathbb{E}[(L_t^a(Z^\theta))^p] da d\theta \\ &\leq 2^p(m\epsilon^2)^{-p} \|f_l\|_\infty^p c_L. \end{aligned}$$

By the monotone convergence theorem, we have $J_l^Y \rightarrow J^Y$ as $l \rightarrow \infty$. the L^p estimate for $J^Y(T)$ in (10b),

$$\mathbb{E}[|J^Y(T)|^p] \leq 2^p(m\epsilon^2)^{-p} \|f\|_\infty^p c_L.$$

On the other hand, let us consider J^σ in (10b). By the construction of φ_m we have

$$\begin{aligned} J^\sigma(t) &= \int_0^t u_m''(Y_n(s)) |\sigma(X(s)) - \sigma_n(X(s))|^2 ds \\ &\leq 2(ma_m)^{-1} \int_0^t |\sigma(X(s)) - \sigma_n(X(s))|^2 ds. \end{aligned}$$

If the convergence rate (6) holds, then using Hölder's inequality we obtain

$$\begin{aligned} &\left(\int_0^t |\sigma(X(s)) - \sigma_n(X(s))|^2 ds\right)^p \\ &\leq \epsilon^{-2p} \left(\int_{\mathbb{R}} |\sigma(a) - \sigma_n(a)| L_t^a(X) da\right)^p \\ &\leq \epsilon^{-2p} \left(\int_{\mathbb{R}} |\sigma(a) - \sigma_n(a)| da\right)^{p-1} \int_{\mathbb{R}} |\sigma(a) - \sigma_n(a)| |L_t^a(X)|^p da \\ &\leq \epsilon^{-2p} (C_0 n^{-1})^{p-1} \int_{\mathbb{R}} |\sigma(a) - \sigma_n(a)| |L_t^a(X)|^p da, \end{aligned}$$

and then we obtain

$$\mathbb{E}\left(\int_0^t |\sigma(X(s)) - \sigma_n(X(s))|^2 ds\right)^p \leq \epsilon^{-2p} (C_0 n^{-1})^{p-1} \times C_0 n^{-1} c_L = \epsilon^{-2p} c_L (C_0 n^{-1})^p,$$

where $L_t^a(X)$ is the local time of X such that

$$\sup_{a \in \mathbb{R}} \mathbb{E}[(L_t^a(Z^0))^p] = \sup_{a \in \mathbb{R}} \mathbb{E}[(L_t^a(X))^p] \leq c_L.$$

On the other hand, the rate of convergence of (7) implies that

$$\left(\int_0^t |\sigma(X(s)) - \sigma_n(X(s))|^2 ds\right)^p \leq (t C_0 n^{-1})^p.$$

Then, under the stability rate condition (6) or (7), the term of J^σ is estimated in L^p as follows:

$$\mathbb{E}[|J^\sigma(T)|^p] \leq 2^p(ma_m)^{-p} (\epsilon^{-2p} c_L + T^p) (C_0 n^{-1})^p.$$

In short, combining these estimates with (10b) we obtain

$$(10f) \quad \mathbb{E}[|J(T)|^p] \leq 2^p m^{-p} c_{JY} + 2^p (ma_m n)^{-p} c_{J\sigma},$$

where we define

$$\begin{aligned} c_{JY} &:= \epsilon^{-2p} \|f\|_\infty^p c_L \\ c_{J\sigma} &:= (\epsilon^{-2p} c_L + T^p) C_0^p. \end{aligned}$$

Now let us estimate Y_n in the case of $p = 1$,

$$|Y_n(t)| \leq a_{m-1} + M(t) + J^\sigma(t) + J^Y(t).$$

Then we obtain

$$(10g) \quad \mathbb{E}[|Y_n(t)|] \leq a_{m-1} + 2c_{JY}m^{-1} + 2c_{J\sigma}(ma_m n)^{-1} = A_{m,n}, \quad \text{say.}$$

Here, in the Yamada-Watanabe method let us choose $(a_m)_{m \in \mathbb{N}}$ as $a_m = \exp(-m(m+1)/2)$, $a_0 = 1$ and select a sequence $(m_n)_{n \in \mathbb{N}}$ such that

$$(10h) \quad \frac{1}{a_{m_n} n} \leq 1$$

holds for $n > 2$. Since we have that $a_{m-1} \leq 1/m$ for any $m \in \mathbb{N}$, there exists a finite positive number c_a such that $A_{m_n, n}$ in (10g) is bounded by

$$(10i) \quad \mathbb{E}[|Y_n(t)|] \leq A_{m_n, n} \leq c_a (\log n)^{-\frac{1}{2}}.$$

We shall obtain the L^p -estimate (8) from this L^1 -estimate (10i). Let us estimate the quadratic process $\langle M \rangle$,

$$\begin{aligned} \langle M \rangle_T &\leq \int_0^t |\sigma(X(s)) - \sigma_n(X_n(s))|^2 ds \\ &\leq \int_0^T |\sigma(X(s)) - \sigma_n(X(s))|^2 ds + \int_0^T |\sigma_n(X(s)) - \sigma_n(X_n(s))|^2 ds \\ &=: \langle M^\sigma \rangle_T + \langle M^Y \rangle_T, \quad \text{say.} \end{aligned}$$

By the same argument as we estimate $\mathbb{E}|J(T)|^p$ in (10f) with $c_{J\sigma}$, we have

$$\mathbb{E}[\langle M^\sigma \rangle_T^{\frac{p}{2}}] \leq \sqrt{\mathbb{E}[\langle M^\sigma \rangle_T^p]} \leq \sqrt{c_{J\sigma} n^{-p}}.$$

On the other hand, for any positive number y we have

$$\begin{aligned} \langle M^Y \rangle_T &= \int_0^T |\sigma_n(X(s)) - \sigma_n(X_n(s))|^2 ds \\ &\leq \int_0^T |f_l(X(s)) - f_l(X_n(s))| 1_{(|Y_n(s)| > y)} ds \\ &\quad + \int_0^T |f_l(X(s)) - f_l(X_n(s))| 1_{(|Y_n(s)| \leq y)} ds \\ &\leq 2\|f\|_\infty \int_0^T 1_{(|Y_n(s)| > y)} ds + y \int_0^T \int_0^1 f'_l(Z^\theta(s)) d\theta ds. \end{aligned}$$

Therefore, putting $y = (\log n)^{-\frac{1}{2(p+1)}}$, there exists a positive number c_M such that

$$\begin{aligned} \mathbb{E}[\langle M^Y \rangle_t^{\frac{p}{2}}] &\leq \sqrt{\mathbb{E}[\langle M^Y \rangle_t^p]} \\ &\leq \sqrt{2^p \max\left((2\|f\|_\infty)^p T^p y^{-1} (\log n)^{-\frac{1}{2}}, y^p c_{JY}\right)} \leq c_M (\log n)^{-\frac{p}{4(p+1)}}. \end{aligned}$$

Then by combining these estimates with (10d) and the Burkholder-Davis-Gundy Inequality we observe that there exists a constant value c_p such that

$$\begin{aligned} & \mathbb{E} \left[\sup_{0 \leq t \leq T} |X(t) - X_n(t)|^p \right] \\ & \leq c_p \left\{ a_{m-1}^p + \mathbb{E}[\langle M^\sigma \rangle_t^{\frac{p}{2}}] + \mathbb{E}[\langle M^Y \rangle_t^{\frac{p}{2}}] + \mathbb{E}[|J^\sigma(T)|^p] + \mathbb{E}[|J^Y(T)|^p] \right\}. \\ & \leq c_p \left\{ a_{m-1}^p + \sqrt{c_{J\sigma} n^{-p}} + \mathbb{E}[\langle M^Y \rangle_t^{\frac{p}{2}}] + 2^p c_{JY} m^{-p} + 2^p c_{J\sigma} (ma_m n)^{-p} \right\}. \end{aligned}$$

Since we have selected the sequence $(m_n)_{n \in \mathbb{N}}$ as in (10h),

$$\begin{aligned} & \mathbb{E} \left[\sup_{0 \leq t \leq T} |X(t) - X_n(t)|^p \right] \\ & \leq c_p (\log n)^{-\frac{p}{4(p+1)}} \{1 + \sqrt{c_{J\sigma}} + c_M + 2^p c_{JY} + 2^p c_{J\sigma}\}, \end{aligned}$$

for $n > 2$. Therefore we obtain the desired result. \square

4. SDES DRIVEN BY SYMMETRIC α -STABLE PROCESSES

In this section, we introduce a symmetric α stable process Z and the sequence of SDEs driven by Z :

$$(11) \quad X(t) = X(0) + \int_0^t \sigma(X(s-)) dZ_s,$$

$$(12) \quad X_n(t) = X_n(0) + \int_0^t \sigma_n(X_n(s-)) dZ_s \quad (n \in \mathbb{N}),$$

where both $\{\sigma_n\}_{n \in \mathbb{N}}$ and σ satisfy the following Belfadli-Ouknine condition.

Definition 4.1. *We say that a function σ satisfies the Belfadli-Ouknine condition if σ satisfies Definition 2.1 (1) and the modified condition (2)' of Definition 2.1: there exists an increasing function f such that*

$$|\sigma(x) - \sigma(y)|^\alpha \leq |f(x) - f(y)| \text{ and } \|f\|_\infty < \infty$$

holds for any $x, y \in \mathbb{R}$.

Under the assumptions, the pathwise uniqueness holds for X and also X_n , $n \in \mathbb{N}$, which is proven by Belfadli and Ouknine in [1]. Moreover, it is shown by Hashimoto [6] that the solution is realized by the stability problem. The local time is again the key to compute the strong convergence rates via stability problem. For the results of the local time for the symmetric α stable process see K. Yamada [15] and Salminen and Yor [11].

Theorem 4.1. *Let α be a positive number with $1 < \alpha < 2$. Assume $X_n(0) \equiv X(0)$ with $\mathbb{E}|X_n(0)|^{\alpha-1} < \infty$. If σ and $\{\sigma_n\}_{n \in \mathbb{N}}$ satisfy the Belfadli-Ouknine condition and either of the stability rate condition (6) or (7), then there exists a positive constant C_2 such that*

$$(13) \quad \mathbb{E} \left[|X(t) - X_n(t)|^{\alpha-1} \right] \leq C_2 (\log n)^{-\frac{\alpha-1}{2}}$$

holds for $0 \leq t \leq T$ and $n > 2$.

Proof. Using Émery's inequality in [4], we have

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} |X(t)|^{\alpha-1} \right] + \sup_{n \in \mathbb{N}} \mathbb{E} \left[\sup_{0 \leq t \leq T} |X_n(t)|^{\alpha-1} \right] < \infty.$$

Define $Y_n(t) := X(t) - X_n(t)$. Let φ_m be a function as defined in Section 2. Define

$$v_m = v * \varphi_m,$$

where $v(x) = |x|^{\alpha-1}$ and $*$ stands for the convolution. Then we have

$$-(a_{m-1})^{\alpha-1} + v_m(x) \leq v(x) \leq (a_{m-1})^{\alpha-1} + v_m(x),$$

holds for $x \in \mathbb{R}$, see [6].

By the Itô formula we have

$$\begin{aligned} |Y_n(t)|^{\alpha-1} &\leq (a_{m-1})^{\alpha-1} + v_m(Y_n(t)) \\ &= (a_{m-1})^{\alpha-1} + M(t) + J(t), \end{aligned}$$

where M and J are defined as follows:

$$\begin{aligned} M(t) &= \int_0^t u'_m(Y_n(s)) \{ \sigma(X(s)) - \sigma_n(X_n(s)) \} dZ_s, \\ J(t) &= K_\alpha \int_0^t \varphi_m(Y_n(s)) |\sigma(X(s)) - \sigma_n(X_n(s))|^\alpha ds, \end{aligned}$$

where we define a constant $K_\alpha = -\Gamma(\alpha) \cos(\alpha\pi/2)/2 \geq 0$ for $\alpha \in (1, 2)$ from the gamma function $\Gamma(\cdot)$.

Now we also define J^σ and J^Y such that

$$\begin{aligned} J(t) &\leq 2K_\alpha \int_0^t \varphi_m(Y_n(s)) \{ \sigma(X(s)) - \sigma_n(X_n(s)) \}^\alpha ds \\ &\quad + 2K_\alpha \int_0^t \varphi_m(Y_n(s)) |\sigma_n(X(s)) - \sigma_n(X_n(s))|^\alpha ds, \\ &=: 2K_\alpha (J^\sigma(t) + J^Y(t)), \text{ say.} \end{aligned}$$

For $\theta \in [0, 1]$, let us define

$$V(t) \equiv V_t \equiv V^\theta(t) := X(t) + \theta(X_n(t) - X(t)).$$

Notice that we have for $x, y \in \mathbb{R}$,

$$f_l(x) - f_l(y) = (x - y) \int_0^1 f'_l(x + \theta(y - x)) d\theta.$$

Then we have that

$$\begin{aligned} J_l^Y(t) &:= \int_0^t \varphi_m(Y_n(s)) |f_l(X(s)) - f_l(X_n(s))| ds \\ &= \int_0^t \varphi_m(Y_n(s)) |Y_n(s)| 1_{(0 < |Y_n(s)|)} \frac{|f_l(X(s)) - f_l(X_n(s))|}{|Y_n(s)|} ds \\ &\leq 2m^{-1} \int_0^t \int_0^1 f'_l(V^\theta(s)) d\theta ds. \end{aligned}$$

Define

$$A_t := \int_0^t |H_s|^\alpha ds \text{ where } H_s := \sigma(X(s)) + \theta(\sigma_n(X_n(s)) - \sigma(X(s)))$$

By the time-change method described in [1] or [13], we have

$$\int_0^t f'_l(V_s) ds = \int_0^t f'_l(\tilde{Z}_{A_s}) |H_s|^{-\alpha} dA_s \leq \epsilon^{-\alpha} \int_0^t f'_l(\tilde{Z}_{A_s}) dA_s,$$

where $V_s =: \tilde{Z}_{A_s}$. Making the random change of variable $s = \tau_u := \inf\{t \geq 0 : A_t > u\}$, we obtain

$$\int_0^t f'_l(V^\theta(s)) ds \leq \epsilon^{-\alpha} \int_0^{A_t} f'_l(\tilde{Z}_u) du \leq \epsilon^{-\alpha} \int_0^{\|\sigma\|_\infty^\alpha T} f'_l(\tilde{Z}_u) du,$$

where \tilde{Z} is the symmetric α stable process with respect to the filtration $\mathcal{G}_t := \mathcal{F}_{\tau^{-1}(t)}$. By the occupation time formula of the local time L_t^a of \tilde{Z} we have

$$\int_0^{\tilde{T}} f'_l(\tilde{Z}_u) du = \int_{-\infty}^{\infty} f'_l(a) L_{\tilde{T}}^a(\tilde{Z}) da,$$

for $\tilde{T} := \|\sigma\|_\infty^\alpha T$, for an example see [2]. Indeed, by the Tanaka-Meyer-K. Yamada formula in [15] implies that

$$\begin{aligned} L_t^a(\tilde{Z}) &= |\tilde{Z}_t - a|^{\alpha-1} - |a|^{\alpha-1} - N_t^a \leq \left| |\tilde{Z}_t - a|^{\alpha-1} - |a|^{\alpha-1} \right| - N_t^a \\ &\leq \left| |\tilde{Z}_t - a| - |a| \right|^{\alpha-1} - N_t^a \\ &\leq |\tilde{Z}_t|^{\alpha-1} - N_t^a, \end{aligned}$$

where N^a is a squared martingale such that

$$\langle N^a \rangle_t = c_\alpha \int_0^t \frac{ds}{|\tilde{Z}_s - a|^{2-\alpha}}.$$

Then we obtain

$$c_{\tilde{L}} := \sup_{a \in \mathbb{R}} \mathbb{E}[L_{\tilde{T}}^a(\tilde{Z})] < \infty.$$

By Jensen's inequality for the concave function $x^{\alpha-1}$, we have

$$\begin{aligned} \mathbb{E}[|J_l^Y(t)|^{\alpha-1}] &\leq 2^{\alpha-1} (m\epsilon^\alpha)^{-(\alpha-1)} \mathbb{E} \left[\left(\int_0^1 \int_{-\infty}^\infty f'_l(a) L_{\tilde{T}}^a(\tilde{Z}) da d\theta \right)^{\alpha-1} \right] \\ &\leq 2^{\alpha-1} (m\epsilon^\alpha)^{-(\alpha-1)} \left(\int_0^1 \int_{-\infty}^\infty f'_l(a) \mathbb{E}[L_{\tilde{T}}^a(\tilde{Z})] da d\theta \right)^{\alpha-1} \\ &\leq 2^{\alpha-1} (m\epsilon^\alpha)^{-(\alpha-1)} (\|f\|_\infty c_{\tilde{L}})^{\alpha-1}. \end{aligned}$$

On the other hand, let us consider J^σ ,

$$\begin{aligned} J^\sigma(t) &= \int_0^t \varphi_m(Y_n(s)) |\sigma(X(s)) - \sigma_n(X(s))|^\alpha ds \\ &\leq 2(ma_m)^{-1} \int_0^t |\sigma(X(s)) - \sigma_n(X(s))|^\alpha ds. \end{aligned}$$

Note that

$$\int_0^t |\sigma(X(s)) - \sigma_n(X(s))|^\alpha ds \leq \|\sigma_0\|_\infty^{\alpha-1} \epsilon^{-\alpha} \int_{\mathbb{R}} |\sigma(a) - \sigma_n(a)| L_t^a(X) da,$$

where $L_t^a(X)$ is the local time of X such that

$$\sup_{a \in \mathbb{R}} \mathbb{E}[L_t^a(Z^0)] = \sup_{a \in \mathbb{R}} \mathbb{E}[L_t^a(X)] \leq c_{\tilde{L}},$$

with $c_{\tilde{L}}$ as in (10e) in Section 3, and we define

$$\|\sigma_0\|_\infty := \sup_{x \in \mathbb{R}} (|\sigma(x)|, |\sigma_n(x)|) < \infty.$$

The stability rate condition (6) or (7) implies that

$$\mathbb{E}[|J^\sigma(t)|^{\alpha-1}] \leq 2^{\alpha-1} (ma_m)^{-(\alpha-1)} \left(t^{\alpha-1} + (\|f\|_\infty c_{\tilde{L}})^{\alpha-1} \right) (\|\sigma_0\|_\infty C_0 n^{-1})^{\alpha-1}.$$

In short, we obtain

$$\mathbb{E}|J(t)|^{\alpha-1} \leq (4K_\alpha)^{\alpha-1} c_{JY} m^{-(\alpha-1)} + (4K_\alpha)^{\alpha-1} c_{J\sigma} (ma_m n)^{-(\alpha-1)},$$

where we define

$$\begin{aligned} c_{JY} &:= (\epsilon^{-\alpha} \|f\|_\infty c_{\tilde{L}})^{\alpha-1}, \\ c_{J\sigma} &:= \left(t^{\alpha-1} + (\|f\|_\infty c_{\tilde{L}})^{\alpha-1} \right) (\|\sigma_0\|_\infty C_0)^{\alpha-1} \end{aligned}$$

Now let us estimate $|Y|^{\alpha-1}$,

$$|Y_n(t)|^{\alpha-1} \leq (a_{m-1})^{\alpha-1} + M(t) + J^\sigma(t) + J^Y(t).$$

Then we obtain

$$\mathbb{E}[|Y_n(t)|^{\alpha-1}] \leq (a_{m-1})^{\alpha-1} + (4K_\alpha)^{\alpha-1} \left(c_{JY} m^{-(\alpha-1)} + c_{J\sigma} (ma_m n)^{-(\alpha-1)} \right) = A_{m,n}, \text{ say.}$$

Here let us consider the one of the sequence $(a_m)_{m \in \mathbb{N}}$ such that $a_m = \exp(-m(m+1)/2)$. Then select a sequence $(m_n)_{n \in \mathbb{N}}$ such that

$$\frac{1}{a_m n} \leq 1$$

holds for for $n > 2$. Since we have that $a_{m-1} \leq 1/m$ for any $m \in \mathbb{N}$, there exists a finite positive number c_a such that

$$\mathbb{E}[|Y_n(t)|^{\alpha-1}] \leq A_{m_n, n} \leq c_a (\log n)^{-\frac{\alpha-1}{2}}.$$

□

5. INVARIANT PROPERTY UNDER REMOVAL DRIFT

In this section, we consider the SDEs of (2) with the drifts of SDEs driven by the Wiener process W_s . Now we introduce the method of removal of the drifts.

Suppose that there exists a strong solution X_n and X to the equation (1) and (2) and taking values on $I \equiv (l, k)$ for $-\infty \leq l < k \leq \infty$. Consider the coefficients b, b_n and σ, σ_n for $n \in \mathbb{N}$ such that the functions $b_n \sigma_n^{-2}$ and $b \sigma^{-2}$ belong to $L^1(I)$, i.e.,

$$(14) \quad \sup_{n \in \mathbb{N}} \int_I |b_n(u) \sigma_n^{-2}(u)| du < \infty \text{ and } \int_I |b(u) \sigma^{-2}(u)| du < \infty.$$

Now let us consider the scale functions given by

$$(15) \quad s'_n(x) := \exp \left(-2 \int^x \frac{b_n(u)}{\sigma_n^2(u)} du \right) \text{ and } s'(x) := \exp \left(-2 \int^x \frac{b(u)}{\sigma^2(u)} du \right).$$

By Itô's formula we have the drift-less processes $\bar{X}_n(t) := s_n(X_n(t))$ taking values on $s_n(I)$ and $\bar{X}(t) := s(X(t))$ taking values on $s(I)$ such that

$$(16) \quad \bar{X}_n(t) - \bar{X}_n(0) = \int_0^t \bar{\sigma}(\bar{X}_n(s)) dW_s$$

$$(17) \quad \bar{X}(t) - \bar{X}(0) = \int_0^t \bar{\sigma}(\bar{X}(s)) dW_s,$$

where

$$\begin{aligned} \bar{\sigma}_n(\bar{x}) &:= (\sigma_n s'_n) \circ s_n^{-1}(\bar{x}) = \sigma_n(s_n^{-1}(x)) s'_n(s_n^{-1}(\bar{x})) \text{ for } \bar{x} \in s_n(I), \\ \bar{\sigma}(x) &:= (\sigma s') \circ s^{-1}(x) = \sigma(s^{-1}(x)) s'(s^{-1}(x)) \text{ for } x \in s(I). \end{aligned}$$

The Nakao-Le Gall condition is invariant under the removal of drifts in the following sense.

Lemma 5.1. *If the diffusion coefficient σ satisfies the Nakao-Le Gall's condition, $\sigma \in \mathcal{C}_{NL}(\epsilon, \|f\|_\infty)$, and the drift coefficient b is bounded and in L^1 , then so does $\bar{\sigma}$ on $s(I)$.*

Proof. By the assumption $\sigma \in \mathcal{C}_{NL}(\epsilon, \|f\|_\infty)$, we have

$$\int_I |b(u)| |\sigma^{-2}(u)| du \leq \epsilon^{-2} \int_I |b(u)| du < \infty.$$

In addition, the boundedness of b implies that for $x \in I$

$$\exp(-2\|b\sigma^{-2}\|_{L^1}) \leq s'(x) \leq \exp(2\|b\sigma^{-2}\|_{L^1}).$$

Then there exists a positive number ϵ' such that

$$0 < \epsilon' \leq \bar{\sigma}(\bar{x})$$

holds for $\bar{x} \in s(I)$.

On the other hand, using the boundedness of b we have

$$|s''(x)| = |(-2b(x)\sigma^{-2}(x))s'(x)| \leq 2\|b\sigma\|_{L^1} \exp(2\|b\sigma^{-2}\|_{L^1}) < \infty$$

where $\|b\sigma\|_{L^1} := \sup_{u \in I} |b(u)\sigma(u)|$. Thus s' is Lipschitz continuous. Therefore there exists a monotone function f_b such that

$$|\bar{\sigma}(\bar{x}) - \bar{\sigma}(\bar{y})| \leq |f_b(x) - f_b(y)|,$$

holds for $x, y \in s(I)$. \square

This result suggests that the stability problem is independent of the drift coefficient itself. To be more precious, these convergence rates are also invariant under the removal drift as follows.

Corollary 5.1 (Invariant property). *Suppose that the diffusion coefficient σ, σ_n for $n \in \mathbb{N}$ satisfy the Nakao-Le Gall's condition, $\sigma, \sigma_n \in \mathcal{C}_{NL}(\epsilon, \|f\|_\infty)$, and the drift coefficient b and b_n are uniformly bounded and in L^1 . If there exists a constant C_0 such that*

$$\int_{x \in \mathbb{R}} |b_n(x) - b(x)| + |\sigma_n(x) - \sigma(x)| dx \leq C_0 n^{-1},$$

then there exists a positive number \bar{C}_0 such that

$$\int_{S_n} |\bar{\sigma}(\bar{x}) - \bar{\sigma}_n(\bar{x})| d\bar{x} \leq \bar{C}_0 n^{-1}.$$

where $S_n = s_n(I) \cap s(I)$.

Proof. Let us observe

$$\begin{aligned} & |\bar{\sigma}_n(\bar{x}) - \bar{\sigma}(\bar{x})| \\ & \leq |\sigma_n(s_n^{-1}(\bar{x}))| (|s'_n(s_n^{-1}(\bar{x})) - s'(s_n^{-1}(\bar{x}))| + |s'(s_n^{-1}(\bar{x})) - s'(s^{-1}(\bar{x}))|) \\ & \quad + |s'(s^{-1}(\bar{x}))| (|\sigma_n(s_n^{-1}(\bar{x})) - \sigma(s_n^{-1}(\bar{x}))| + |\sigma(s_n^{-1}(\bar{x})) - \sigma(s^{-1}(\bar{x}))|) \\ & \leq 2\|f\|_\infty (|s'_n(s_n^{-1}(\bar{x})) - s'(s_n^{-1}(\bar{x}))| + |s'(s_n^{-1}(\bar{x})) - s'(s^{-1}(\bar{x}))|) \\ & \quad + \|s'(s^{-1})\|_\infty (|\sigma_n(s_n^{-1}(\bar{x})) - \sigma(s_n^{-1}(\bar{x}))| + |\sigma(s_n^{-1}(\bar{x})) - \sigma(s^{-1}(\bar{x}))|) \end{aligned}$$

By the definition of the scale functions s_n and s we have

$$\begin{aligned} s(x) - s_n(x) &= \int^x s'(y) - s'_n(y) dy \\ &= \int^x \exp\left(-2 \int^y b(u)\sigma^{-2}(u) du\right) dy - \exp\left(-2 \int^y b_n(u)\sigma_n^{-2}(u) du\right) dy \\ &= \int^x (\log s'(y) - \log s'_n(y)) \left(\int_0^1 \exp(\theta \log s'(y) + (1-\theta) \log s'_n(y)) d\theta\right) dy \\ &= \int^x \left(\log \frac{s'(y)}{s'_n(y)}\right) \left(\int_0^1 (s'(y))^\theta (s'_n(y))^{1-\theta} d\theta\right) dy. \end{aligned}$$

Now put c_{F1} and c_{F2} as follows:

$$\begin{aligned} c_{s1} &:= \min \left\{ \exp\left(-2 \sup_{n \in \mathbb{N}} \|b_n \sigma_n^{-2}\|_{L^1}\right), \exp\left(-2 \|b \sigma^{-2}\|_{L^1}\right) \right\}, \\ c_{s2} &:= \max \left\{ \exp\left(2 \sup_{n \in \mathbb{N}} \|b_n \sigma_n^{-2}\|_{L^1}\right), \exp\left(2 \|b \sigma^{-2}\|_{L^1}\right) \right\}. \end{aligned}$$

By the assumption, we have

$$\begin{aligned} \left| \log \frac{s'(y)}{s'_n(y)} \right| &\leq \int_I |b_n(u)\sigma^2(u) - \sigma_n^2(u)b(u)| \sigma^{-2}(u) \sigma_n^{-2}(u) du \\ &\leq \int_I |b_n(u) - b(u)| \sigma_n^{-2}(u) + |\sigma^2(u) - \sigma_n^2(u)| |b(u)| \sigma^{-2}(u) \sigma_n^{-2}(u) du. \end{aligned}$$

Then by the rate of the stability convergence rate condition we have

$$\sup_{x \in \mathbb{R}} |s'(x) - s'_n(x)| \leq c_{s2} c_{b,\sigma} n^{-1},$$

where $c_{b,\sigma} := \|\sup_{n \in \mathbb{N}} n \log(s'/s'_n)\|_{L^1} < \infty$.

For $\bar{x} = s^{-1}(y)$ ($y \in s(I)$) and $\bar{x} = s_n^{-1}(y_n)$ ($y_n \in s_n(I)$) we have

$$(s^{-1})'(\bar{x}) - (s_n^{-1})'(\bar{x}) = \frac{1}{s'(y)} - \frac{1}{s'_n(y_n)} = \frac{s'_n(y_n) - s'(y)}{s'(y)s'_n(y_n)}.$$

Then it implies that

$$\sup_{\bar{x} \in \mathbb{R}} |(s^{-1})'(\bar{x}) - (s_n^{-1})'(\bar{x})| \leq c_{s2} c_{s1}^{-2} c_{b,\sigma} n^{-1} =: c'_{b,\sigma} n^{-1}.$$

Since we have

$$s'(s_n^{-1}(\bar{x})) - s'(s^{-1}(\bar{x})) = (s_n^{-1}(\bar{x}) - s^{-1}(\bar{x})) \left(\int_0^1 (-2b\sigma^{-2}s')(\theta s_n^{-1}(\bar{x}) + (1-\theta)s^{-1}(\bar{x})) d\theta \right),$$

then we obtain that

$$\begin{aligned} & \int_{S_n} |s'(s_n^{-1}(\bar{x})) - s'(s^{-1}(\bar{x}))| d\bar{x} \\ &= \int_{S_n} |s_n^{-1}(\bar{x}) - s^{-1}(\bar{x})| \left| \int_0^1 (-2b\sigma^{-2}s')(\theta s_n^{-1}(\bar{x}) + (1-\theta)s^{-1}(\bar{x})) d\theta \right| d\bar{x} \\ &\leq 2n^{-1} c'_{b,\sigma} \times \int_0^1 \int_{S_n} |(b\sigma^{-2})(\theta s_n^{-1}(\bar{x}) + (1-\theta)s^{-1}(\bar{x}))| d\bar{x} d\theta \\ &\leq 2n^{-1} c'_{b,\sigma} \times \int_0^1 \int_{S_n} |(b\sigma^{-2})(g_\theta(\bar{x}))| g'_\theta(\bar{x}) d\bar{x} d\theta \\ &\leq 2n^{-1} c'_{b,\sigma} \|b\sigma^{-2}\|_{L^1(I)} < \infty \end{aligned}$$

where we define

$$g_\theta(\bar{x}) := \theta s_n^{-1}(\bar{x}) + (1-\theta)s^{-1}(\bar{x})$$

and note that $c_{s2}^{-1} \leq g'_\theta(\bar{x}) \leq c_{s1}^{-1}$.

Considering the transformation of variable $y_n = s_n^{-1}(\bar{x})$, we have

$$\int_{S_n} |\sigma_n(s_n^{-1}(\bar{x})) - \sigma(s^{-1}(\bar{x}))| d\bar{x} \leq c_{s2} \int_{S_n} |\sigma_n(y_n) - \sigma(y_n)| dy_n \leq c_{s2} C_0 n^{-1}.$$

Using the sequence of $\{f_l\}_{l \in \mathbb{N}}$ in Remark 2.2, we have

$$\begin{aligned} & \int_{S_n} |\sigma(s_n^{-1}(\bar{x})) - \sigma(s^{-1}(\bar{x}))|^2 d\bar{x} \leq \int_{S_n} |f_l(s_n^{-1}(\bar{x})) - f_l(s^{-1}(\bar{x}))| dx \\ &= \int_{S_n} |s_n^{-1}(\bar{x}) - s^{-1}(\bar{x})| \left| \int_0^1 f'_l(\theta s_n^{-1}(\bar{x}) + (1-\theta)s^{-1}(\bar{x})) d\theta \right| d\bar{x} \\ &\leq c'_{b,\sigma} n^{-1} \int_0^1 \int_{S_n} c_{s2} g'_\theta(\bar{x}) f'_l(g_\theta(\bar{x})) d\bar{x} d\theta. \\ &\leq c'_{b,\sigma} n^{-1} \|f\|_\infty. \end{aligned}$$

Therefore we obtain

$$\begin{aligned} & \int_{S_n} |\bar{\sigma}_n(\bar{x}) - \bar{\sigma}(\bar{x})| d\bar{x} \\ &\leq (\|\sigma_n\|_{S_n, \infty} (c_{s2} c_{b,\sigma} + 2c'_{b,\sigma} \|b\sigma^{-2}\|_{L^1(I)}) + c_{s2} (c_{s2} C_0 + c'_{b,\sigma} \|f\|_{S_n, \infty})) n^{-1}, \end{aligned}$$

and hence we obtain the invariant property. \square

In short, we obtain the extended results of Theorem 3.1 with the drifts.

Theorem 5.1. *Let $T > 0$ and $p \geq 1$ and let X_n for $n \in \mathbb{N}$ be solutions of (1) and X be a solution of (2) such that $\mathbb{E}|X(0)|^p < \infty$ and $X_n(0) \equiv X(0)$. Suppose that σ and $\{\sigma_n\}_{n \in \mathbb{N}}$ satisfy the Nakao-Le Gall condition $\sigma, \sigma_n \in \mathcal{C}_{NL}(\epsilon, \|f\|_\infty)$.*

If the stability rate conditions holds, i.e. there exists a positive constant C_0 such that

$$\int_{x \in \mathbb{R}} |b_n(x) - b(x)| + |\sigma_n(x) - \sigma(x)| dx \leq C_0 n^{-1},$$

Then there exist positive constants C_p ($p \geq 1$) such that

$$\mathbb{E}[|X(t) - X_n(t)|] \leq C_1 (\log n)^{-\frac{1}{2}} \quad (0 \leq t \leq T)$$

and

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} |X(t) - X_n(t)|^p \right] \leq C_p (\log n)^{-\frac{p}{4(p+1)}}$$

holds for $p > 1$ and $n > 2$.

Remark 5.1. *Assume $\mathbb{E}[\sup_{0 \leq t \leq T} |X(t)|^2] < \infty$. This implies that $\mathbb{E}[\int_{\mathbb{R}} L_T^a(X) da] = \mathbb{E}[(X)]_T < \infty$, and then the same convergence rate is obtained under the stability rate of (6).*

REFERENCES

1. R. Belfadli and Y. Ouknine, *On the pathwise uniqueness of solutions of stochastic differential equations driven by symmetric stable Lévy processes*, Stochastics **80** (2008), no. 6, 519–524. MR 2460247 (2010b:60166)
2. Jean Bertoin, *Lévy processes*, Cambridge Tracts in Mathematics, vol. 121, Cambridge University Press, Cambridge, 1996.
3. Griselda Deelstra and Freddy Delbaen, *Convergence of discretized stochastic (interest rate) processes with stochastic drift term.*, Applied Stochastic Models and Data Analysis **14** (1998), no. 1, 77–84.
4. Michel Emery, *Stabilité des solutions des équations différentielles stochastiques application aux intégrales multiplicatives stochastiques*, Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete **41** (1978), no. 3, 241–262 (French).
5. István Gyöngy and Miklós Rásonyi, *A note on Euler approximations for SDEs with Hölder continuous diffusion coefficients.*, Stochastic Processes Appl. **121** (2011), no. 10, 2189–2200.
6. Hiroya Hashimoto, *Approximation and stability of solutions of sdes driven by a symmetric α stable process with non-lipschitz coefficients*, Séminaire de Probabilités XLV (Catherine Donati-Martin, Antoine Lejay, and Alain Rouault, eds.), Lecture Notes in Mathematics, Springer International Publishing, 2013, pp. 181–199 (English).
7. Hiroshi Kaneko and Shintaro Nakao, *A note on approximation for stochastic differential equations.*, Séminaire de probabilités de Strasbourg **22** (1988), 155–162.
8. Shigetoku Kawabata and Toshio Yamada, *On some limit theorems for solutions of stochastic differential equations.*, Séminaire de probabilités XVI, Univ. Strasbourg 1980/81, Lect. Notes Math. 920, 412–441 (1982)., 1982.
9. Jean-François Le Gall, *Applications du temps local aux équations différentielles stochastiques unidimensionnelles.*, Séminaire de probabilités de Strasbourg **17** (1983), 15–31 (French).
10. Shintaro Nakao, *On the pathwise uniqueness of solutions of one-dimensional stochastic differential equations.*, Osaka J. Math. **9** (1972), 513–518.
11. Paavo Salminen and Marc Yor, *Tanaka formula for symmetric lévy processes*, Séminaire de Probabilités XL (Catherine Donati-Martin, Michel Émery, Alain Rouault, and Christophe Stricker, eds.), Lecture Notes in Mathematics, vol. 1899, Springer Berlin Heidelberg, 2007, pp. 265–285.
12. Daniel W. Stroock and S. R. Srinivasa Varadhan, *Multidimensional diffusion processes*, Die Grundlehren der mathematischen Wissenschaften, Springer-Verlag, Berlin, Heidelberg, New York, 1979.
13. Takahiro Tsuchiya, *On the pathwise uniqueness of solutions of stochastic differential equations driven by multi-dimensional symmetric α stable class*, J. Math. Kyoto Univ. **46** (2006), no. 1, 107–121. MR 2260819 (2007e:60050)
14. Shinzo Watanabe and Toshio Yamada, *On the uniqueness of solutions of stochastic differential equations. II.*, J. Math. Kyoto Univ. **11** (1971), 553–563.
15. Keigo Yamada, *Fractional derivatives of local times of α -stable Lévy processes as the limits of occupation time problems.*, Limit theorems in probability and statistics. Fourth Hungarian colloquium on limit theorems in probability and statistics, Balatonlelle, Hungary, June 28–July 2, 1999. Vol. II, Budapest: János Bolyai Mathematical Society, 2002, pp. 553–573 (English).
16. Toshio Yamada, *Sur une construction des solutions d'équations différentielles stochastiques dans le cas non-lipschitzien.*, Séminaire de Probabilités XII (C. Dellacherie, P.A. Meyer, and M. Weil, eds.), Lecture Notes in Mathematics, vol. 649, Springer Berlin Heidelberg, 1978, pp. 114–131 (French).
17. Toshio Yamada and Shinzo Watanabe, *On the uniqueness of solutions of stochastic differential equations.*, Journal of Mathematics of Kyoto University **11** (1971), 155–167.

(H. Hashimoto) SANWA KAGAKU KENKYUSHO CO., LTD.

E-mail address: `hiro_hashimoto@skk-net.com`

(T. Tsuchiya) SCHOOL OF COMPUTER SCIENCE AND ENGINEERING, THE UNIVERSITY OF AIZU

E-mail address, Corresponding author: `suci@probab.com`